PP36697. Proposed by Mihaly Bencze.

In any triangle ABC holds:

1) $\sum r_a r_b (r_a + r_b) \geq 2r \sum r_a^2 + 4s^2 r$;

2) $\sum h_a h_b (h_a + h_b) \geq 2r \sum h_a^2 + \dfrac{8s^2r^2}{R}$.

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1. Since* $\sum r_a r_b = s^2, \sum r_a = 4R + r, \sum r_a^2 = (4R+r)^2 - 2s^2, r_a r_b r_c = \prod \dfrac{sr}{s-a} = \dfrac{s^3r^3}{r^2 s} = s^2 r$ then

$\sum r_a r_b (r_a + r_b) = \sum r_a \cdot \sum r_a r_b - 3r_a r_b r_c = (4R+r)s^2 - 3s^2 r = 2s^2(2R-r)$,

$2r \sum r_a^2 + 4s^2 r = 2r\left((4R+r)^2 - 2s^2\right) + 4s^2 r = 2r(4R+r)^2$

and, therefore,

$\sum r_a r_b (r_a + r_b) \geq 2r \sum r_a^2 + 4s^2 r \Leftrightarrow 2s^2(2R-r) \geq 2r(4R+r)^2 \Leftrightarrow$

(**1**) $\quad s^2(2R-r) \geq r(4R+r)^2$.

Since $s^2 \geq 16Rr - 5r^2$ (Gerretsen's Inequality) and $R \geq 2r$ (Euler's inequality) we obtain $s^2(2R-r) - r(4R+r)^2 \geq (16Rr - 5r^2)(2R-r) - r(4R+r)^2 = 2r(R-2r)(8R-r) \geq 0$.

* Since $r_x = \dfrac{sr}{s-x}$, $x \in \{a,b,c\}$ then $\sum r_a r_b = s^2r^2 \sum \dfrac{1}{(s-a)(s-b)} = \dfrac{s^2r^2}{(s-a)(s-b)(s-c)} \sum (s-c) = \dfrac{s^3r^2}{sr^2} = s^2$, $\sum r_a = \sum \dfrac{sr}{s-a} = \dfrac{sr}{(s-a)(s-b)(s-c)} \sum (s-b)(s-c) = \dfrac{sr(ab+bc+ca-s^2)}{sr^2} = \dfrac{4Rr+r^2}{r} = 4R + r$ and, therefore, $\sum r_a^2 = (4R+r)^2 - 2s^2$.

2. Since $h_a = \dfrac{bc}{2R}$, $h_b = \dfrac{ca}{2R}, h_c = \dfrac{ab}{2R}$ then

$$\sum h_a h_b (h_a + h_b) = \sum \frac{bc}{2R} \cdot \frac{ca}{2R}\left(\frac{bc}{2R} + \frac{ca}{2R}\right) =$$

$$\frac{abc}{8R^3} \sum c^2(a+b) = \frac{abc}{8R^3} \sum ab(a+b) = \frac{abc(\sum a \cdot \sum ab - 3abc)}{8R^3},$$

$$2r \sum h_a^2 + \frac{8s^2r^2}{R} = 2r \sum \frac{b^2c^2}{4R^2} + \frac{8s^2r^2}{R} = \frac{r\left((\sum ab)^2 - 2abc \sum a\right)}{2R^2} + \frac{8R^2s^2r^2}{R^3}$$

and, therefore, $\sum h_a h_b (h_a + h_b) \geq 2r \sum h_a^2 + \dfrac{8s^2r^2}{R} \Leftrightarrow$

$$\frac{abc(\sum a \cdot \sum ab - 3abc)}{8R^3} \geq \frac{r\left((\sum ab)^2 - 2abc \sum a\right)}{2R^2} + \frac{8R^2s^2r^2}{R^3} \Leftrightarrow$$

$$abc(\sum a \cdot \sum ab - 3abc) \geq 4Rr\left((\sum ab)^2 - 2abc \sum a\right) + 64R^2s^2r^2 \Leftrightarrow$$

$$4Rrs(\sum a \cdot \sum ab - 3abc) \geq 4Rr\left((\sum ab)^2 - 2abc \sum a\right) + 64R^2s^2r^2 \Leftrightarrow$$

$$s(\sum a \cdot \sum ab - 3abc) \geq \left((\sum ab)^2 - 4abcs\right) + 4abcs \Leftrightarrow$$

(**2**) $\quad s(\sum a \cdot \sum ab - 3abc) \geq (\sum ab)^2$.

Let $x := s-a, y := s-b, z := s-c, p := xy + yz + zx, q := xyz$. Then, assuming $s = 1$ (due homogeneity), we obtain $x,y,z > 0, x+y+z = 1, a = 1-x, b = 1-y, c = 1-z$, $\sum ab = 1 + p$, $abc = p - q, s(\sum a \cdot \sum ab - 3abc) = 2(1+p) - 3(p-q) = 2 - p + 3q$,

and inequality (**2**) becomes $2-p+3q \ge (1+p)^2 \Leftrightarrow 1-3p-p^2+3q \ge 0.$
Since $3p \le 1$ and $9q \ge 4p-1,$ respectively, $3\sum xy \le (\sum x)^2$ and $\sum x(x-y)(x-z) \ge 0$ (Schure's inequality in p,q notation and normalized by $x+y+z=1$) then
$1-3p-p^2+3q \ge 1-3p-p^2+3 \cdot \dfrac{4p-1}{9} = \dfrac{1}{3}(p+2)(1-3p) \ge 0$).